MEM6804 Modeling and Simulation for Logistics & Supply Chain 物流与供应链建模与仿真

Theory Analysis

Lecture 5: Input Modeling

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  - *"Garbage in, garbage out."*
- *"All models are wrong, but some are useful."* – George Box.
  - There is no "true" model for any stochastic input.
  - The best we can do is to obtain an approximation that yields reasonable and useful results.

Introduction

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  - can capture the physical properties of the system;
  - can be easily tuned to the situation at hand;
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  - can capture the physical properties of the system;
  - can be easily tuned to the situation at hand;
  - can be efficiently generated with certain random variate generation technique.
- Input modeling is sometimes more of an art than an engineering.
  - It nearly always requires the analysts to use their judgment as well as to apply appropriate statistical tools.
  - Since there is no "true" model, it is sensible to run the simulation with several plausible input models to see if the conclusions are robust or highly sensitive to the choices.

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5. If the fit is not good, select another candidate and go to Step 3, or use an empirical distribution.

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Data Collection

- People often have the **false** impression that data are readily available, but it is one of the most *challenging* task in solving a real problem.
- **Never** trust data blindly!
  - A common mistake is to simply throw data into a software and ask for a "best" fit model.
  - Always take into account under what context (e.g., time, potential influence of other factors) the data was collected.

Data Collection

- The collected data can be
  - stale (out of date);
  - "dirty" (containing errors);
  - unexpected;
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- Sometimes the effort or cost to transform data into a usable form, or "clean" data, can be as significant as that required to obtain them.

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  - Check for autocorrelation.
  - Collect input data, not output data.

    – Example: customer arrival times and service times are input, whereas waiting times are output.

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- Do not ignore the physical characteristics of the process when selecting distributions.
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  - Is it bounded or is there no natural bound?
- There are literally hundreds of probability distributions that have been created; many were created with some specific physical process in mind.

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  - **empirical distribution**: Often used when no theoretical distribution seems appropriate.

- The CDF of the **empirical distribution** (empirical CDF) is defined as

$$F_n(x) = \frac{\text{number of points} \le x}{n} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{x_i \le x\}}.$$

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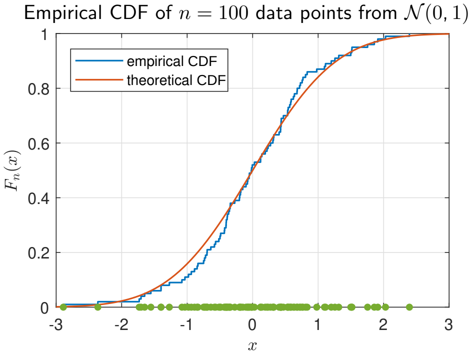

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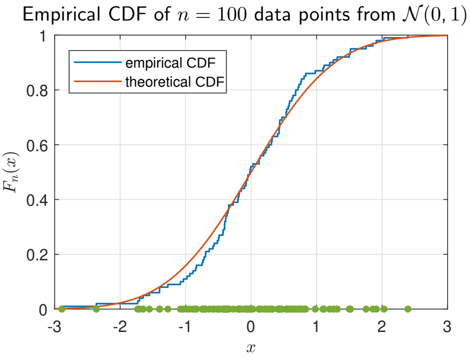

- The empirical CDF is a right-continuous step function.

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    – *Example 1*: the times between the arrivals from a large population of potential customers who act independently.
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  - **Weibull**: Models the time to failure for components.
    – *Note*: the failure rate can be increasing, decreasing, or constant (reduce to exponential distribution).

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  - **Erlang**: Models the time that can be viewed as the sum of several exponentially distributed times.
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    - *Note*: can be shifted away from 0 by adding a constant; can cover a range different from $[0, 1]$ by multiplying by a constant.
  - **triangular**: Models a process for which only the minimum, most likely, and maximum values of the distribution are known.
    - *Example*: only the minimum, most likely, and maximum time required to test a product are known.

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- For continuous data:
  - Corresponds to the pdf of a theoretical distribution.
  - In terms of the *shape*, not the exact *value*!
- For discrete data:
  - Corresponds to the pmf of a theoretical distribution.
  - In terms of both the *shape* and *value* (if the histogram uses relative frequency).
  - If there are few data points, it could be necessary to combine adjacent cells to eliminate the ragged appearance of the histogram.

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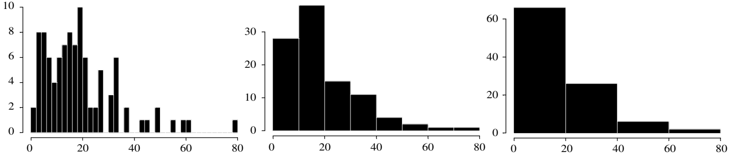

Figure: Ragged, Appropriate and Coarse Histograms (*from Banks et al. (2010)*)

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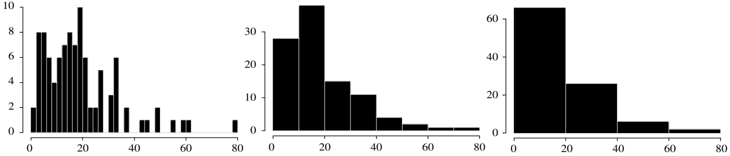

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- Choosing the number of intervals approximately equal to the square root of the sample size often works well in practice (Hines et al. 2002).

Distribution Fitting

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- After a family of distributions has been selected, the next step is to determine the parameters of the distribution that can "best" fit the data.
  - Called distribution fitting, or parameter estimation.
- There are many different approaches and we discuss two simple ones:
  - method of moments (MoM)
  - maximum likelihood estimation (MLE)

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- Suppose the considered distribution family has s unknown parameters.
  1. Analytically compute $\mathbb{E}[X^1], \ldots, \mathbb{E}[X^s]$, as functions of those parameters.
     - *Note*: the moments of common distributions are well-known.
  2. Compute $m_1, \ldots, m_s$ from the data.
  3. Solve $\mathbb{E}[X^k] = m_k, \; k = 1, \ldots, s$, for s unknown parameters.

- Example 1: Suppose $X_1, \ldots, X_n$ are iid from $\mathrm{Gamma}(\alpha, \lambda)$ (in shape & rate parametrization).
  - Recall: $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $\mathbb{E}[X] = \alpha/\lambda$, $\mathrm{Var}(X) = \alpha/\lambda^2$.

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  Estimate α and λ using MoM.

  Solution. The first two moments are

$$\mathbb{E}[X] = \alpha/\lambda = m_1,$$
$$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = (\alpha + \alpha^2)/\lambda^2 = m_2.$$

  Solving two equations yields MoM estimators

$$\widehat{\alpha} = \frac{m_1^2}{m_2 - m_1^2}, \quad \widehat{\lambda} = \frac{m_1}{m_2 - m_1^2}. \quad \blacksquare$$

- Example 2: Suppose $X_1, \ldots, X_n$ are iid from $\mathrm{Exp}(\lambda)$. Estimate λ using MoM.

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  Solution. The first moment is

  $$\mathbb{E}[X] = 1/\lambda = m_1.$$

  So the MoM estimator of λ is $\widehat{\lambda} = \frac{1}{m_1} = \frac{n}{X_1 + \cdots + X_n}$. ■

- Example 3: Suppose $X_1, \ldots, X_n$ are iid from $\mathcal{N}(\mu, \sigma^2)$. Estimate μ and σ^2 using MoM.

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  Solving two equations yields MoM estimators

$$\widehat{\mu} = m_1, \quad \widehat{\sigma}^2 = m_2 - m_1^2. \qquad \blacksquare$$

- Remark: $\widehat{\mu} = \frac{\sum_{i=1}^n X_i}{n}$, and

$$\widehat{\sigma}^2 = \frac{X_1^2 + \cdots + X_n^2}{n} - \bar{X}^2 = \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{n}$$
$$= \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}. \quad \text{(why?)}$$

- Many common distributions have no more than 2 parameters: $\mathrm{Ber}(p)$, $\mathrm{B}(n, p)$, $\mathrm{NB}(r, p)$, $\mathrm{Geo}(p)$, $\mathrm{Pois}(\lambda)$, $\mathrm{Unif}[a, b]$, $\mathrm{Exp}(\lambda)$, $\mathrm{Erl}(k, \lambda)$, $\mathrm{Gamma}(\alpha, \lambda)$, $\mathrm{Beta}(\alpha, \beta)$, $\mathrm{Weibull}(\alpha, \beta)$, $\mathcal{N}(\mu, \sigma^2)$, t_p, χ^2_p.

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- Instead of using MoM, another convenient way to estimate the parameters is using sample mean $\bar{X}$ and sample variance S^2:

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n} = m_1,$$
$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1} = \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{n-1} = \frac{n}{n-1}(m_2 - m_1^2),$$

to solve $\mathbb{E}[X] = \bar{X}$, and $\mathrm{Var}(X) = S^2$ (if necessary).

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to solve $\mathbb{E}[X] = \bar{X}$, and $\text{Var}(X) = S^2$ (if necessary).

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- Many common distributions have no more than 2 parameters: $\mathrm{Ber}(p)$, $\mathrm{B}(n,p)$, $\mathrm{NB}(r,p)$, $\mathrm{Geo}(p)$, $\mathrm{Pois}(\lambda)$, $\mathrm{Unif}[a,b]$, $\mathrm{Exp}(\lambda)$, $\mathrm{Erl}(k,\lambda)$, $\mathrm{Gamma}(\alpha,\lambda)$, $\mathrm{Beta}(\alpha,\beta)$, $\mathrm{Weibull}(\alpha,\beta)$, $\mathcal{N}(\mu,\sigma^2)$, t_p, χ^2_p.

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- Remarks:
  - If $X_1, \dots, X_n$ haven't been observed, $\lambda^* = n/(X_1 + \cdots + X_n)$.
  - The estimator is the same as in MoM.

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- For discrete distributions, replace the pdf with pmf.

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- Try more than one plot/test before making conclusion.

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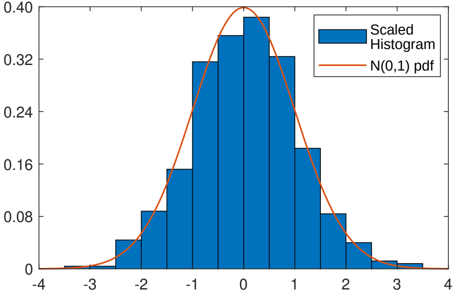

Figure: Example of Scaled Histogram vs. Fitted pdf

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- The q-quantile of X is that value γ such that $\mathbb{P}(X \leq \gamma) = F(\gamma) = q$, for $0 < q < 1$. When $F(x)$ has an inverse, we can write $\gamma = F^{-1}(q)$.
  - Median: 50% quantile.
  - In financial risk management, quantile of the profit-and-loss of a portfolio is also called Value-at-Risk (VaR).

- To make Q-Q plots, given the data $\{x_1, \ldots, x_n\}$ and the fitted distribution with CDF $F(x)$:
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$$y_j \approx F^{-1}\left(\frac{j-0.5}{n}\right),$$

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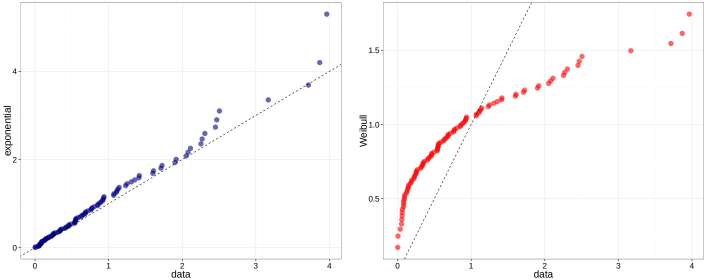

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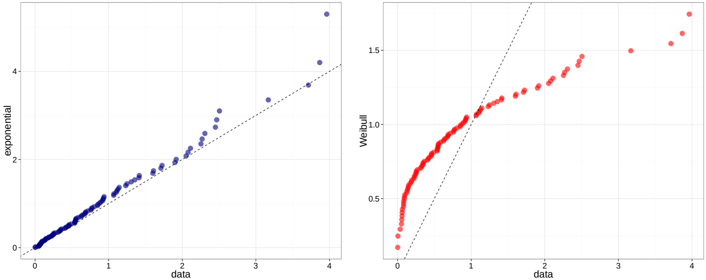

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- The observed values will never fall exactly on a straight line
- The ordered values are not independent because they are ranked. Hence, if one point lies above the line, it is likely that the next one will too.
- The values at the extremes have a much higher variance than those in the middle. So greater discrepancies can be acceptable at the extremes; linearity in the middle is much more important.

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- A hypothesis test is a data-based rule to decide between the null hypothesis (H_0) and the alternative hypothesis (H_1).
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Truth \ Decision	reject H_0	fail to reject H_0
H_0 is true	type I error	correct
H_1 is true	correct	type II error

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  - p-value is smaller than some specified α, or, equivalently,
  - the computed test statistic falls in certain range (called *rejection region*), which is determined by α.

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- Logic: Assume H_0 is true, is it likely to observe the data at hand?
  - If the likelihood is very small (i.e., p-value is very small), then H_0 is unlikely to be true (reject H_0);
  - otherwise, there is no enough evidence to reject H_0 (fail to reject H_0).

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     – Reason: A large value of R indicates a poor fit, whereas a small value indicates a good fit.
     – Question: How large is too large? (i.e., what is the rejection region?)

- View the test statistic R as a random variable.
  - Since we assume the collected data is one observed random sample from some unknown distribution, if we conduct the study multiple times, the values of the statistics will be different because the collected data will be different.
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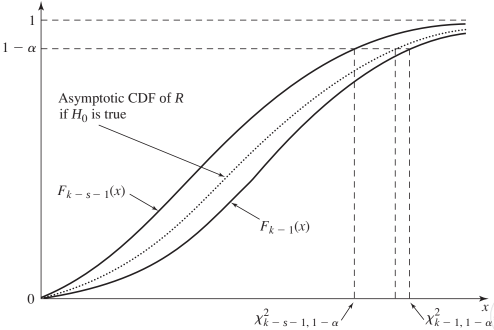

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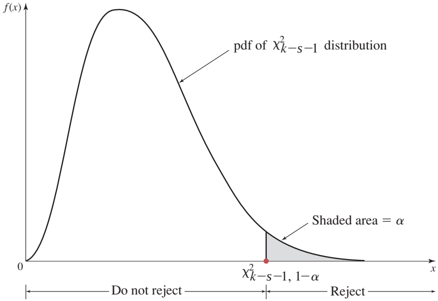

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- In the absence of a definitive guideline for choosing the intervals, it's usually recommended to make E_i equal (or approximately equal) and no less than 5, for all intervals.

- The **Kolmogorov-Smirnov test** (K-S test, 柯尔莫哥洛夫–斯米尔诺夫检验) formally compares the empirical CDF $F_n(x)$ with the CDF of the hypothesized distribution, $F(x)$.

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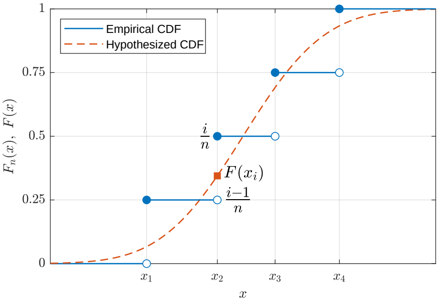

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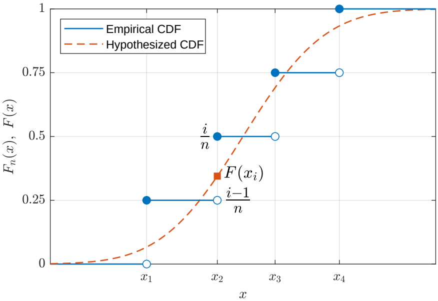

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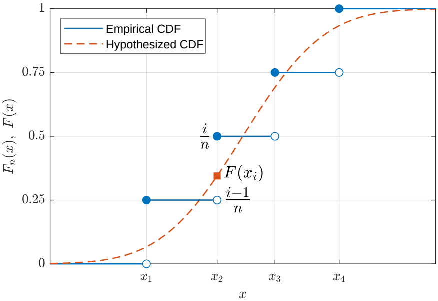

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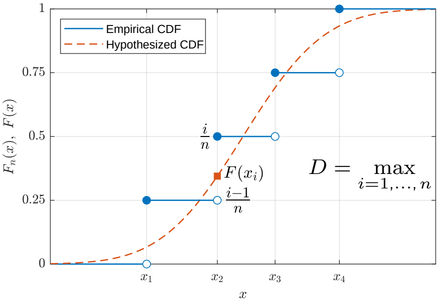

$$D = \max_{i=1,\dots,n} \left\{ \left(\frac{i}{n} - F(x_i) \right) \bigvee \left(F(x_i) - \frac{i-1}{n} \right) \right\}.$$

Note: $x_1, \dots, x_n$ are the **sorted** data points.

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- Advantage of K-S test:
  - It does not require us to group the data in any way, so no information is lost and no troublesome selection is faced.
  - It is valid (exactly) for any sample size, whereas chi-square test is valid only in an asymptotic sense.
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- K-S test is relatively more convenient to be used in a case where the hypothesized distribution is continuous and no parameter is estimated. For example:
  - Test random number generators.
  - Test a Poisson process (more details later).

- Comments on p-value:
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  - Different statistical tests may give different p-values.
  - Whether or not you reject H_0 also depends on the significance level α chosen by yourself.

- Comments on general goodness-of-fit tests:
  - If very little data are available, then a goodness-of-fit test is unlikely to reject any candidate distribution.
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    – H_0 is virtually never exactly true, and even a tiny departure from the hypothesized distribution will be detected for large n.
  - Do not have blind faith in goodness-of-fit tests!
    – Failing to reject a candidate distribution should be taken as only **one piece of evidence** in favor of that choice.
    – Rejecting a candidate distribution should be taken as only **one piece of evidence** against the choice.

- Graphical Methods vs. Statistical Tests
  - Graphical methods *qualitatively* measure the fitting goodness, while statistical tests *quantitatively* measure the fitting goodness.
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- Always combine statistical test results with graphical analysis.
- When no model fits the data satisfactorily, we may end up with the empirical distribution.

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  - It recommends the "best" distribution in its library based on summary measure like the p-value (and perhaps other factors such as discrete or continuous, bounded or unbounded).
- Always keep the following in mind when using such an option:
  - The software might know nothing about the physical basis of the data.
  - Automated best-fit procedures tend to choose the more flexible distributions (gamma over Erlang, Weibull over exponential).
  - But, close conformance to the data does not always lead to the most appropriate input model (overfitting).
  - The limitation of summary measure like p-value.
  - View the automated distribution selection as one suggestion, inspect it using graphical methods, and remember that *the final choice is yours*.

- All the graphical methods and statistical tests can be used to check the uniformity of a random number generator (RNG).
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        - Given $N(T) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as n independent RVs from $\mathrm{Unif}(0,T)$ that are **sorted**.

An Illustrative Example

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1. Data Collection.
   - Perform life tests on a random sample ($n = 50$) of electronic components and record their lifetime, in days:

79.919	3.081	0.062	1.961	5.845
3.027	6.505	0.021	0.013	0.123
6.769	59.899	1.192	34.760	5.009
18.387	0.141	43.565	24.420	0.433
144.695	2.663	17.967	0.091	9.003
0.941	0.878	3.371	2.157	7.579
0.624	5.380	3.148	7.078	23.960
0.590	1.928	0.300	0.002	0.543
7.004	31.764	1.005	1.147	0.219
3.217	14.382	1.008	2.336	4.562

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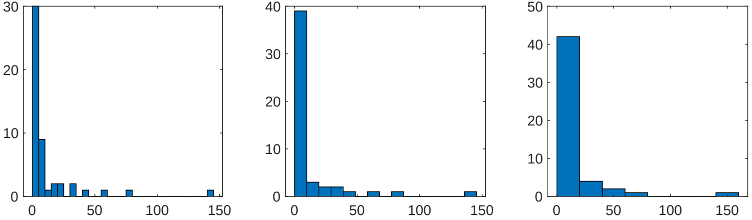

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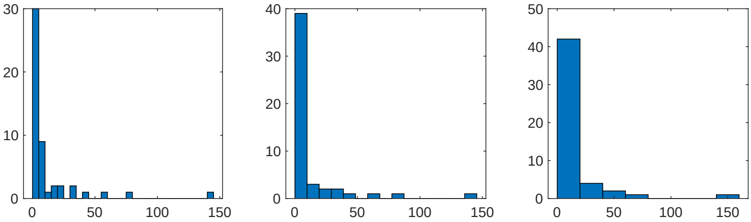

- We decide to first try exponential distribution family $\mathrm{Exp}(\lambda)$.

An Illustrative Example

3. Distribution Fitting.
   - Recall Example 2, MoM (or its variation) and MLE yield the same estimator for λ, which is $\widehat{\lambda} = \frac{n}{X_1 + \cdots + X_n}$.
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3 Distribution Fitting.

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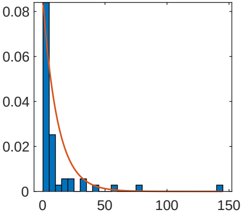
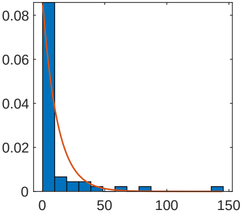
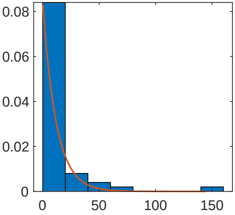
4 Goodness of Fit.

- Scaled histogram vs. pdf of $\mathrm{Exp}(0.084)$.

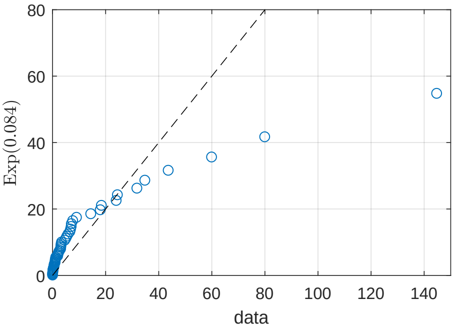






An Illustrative Example

4. Goodness of Fit.
   - Q-Q plot.



An Illustrative Example

4. Goodness of Fit.
   - Chi-square test (H_0: The data come from $\mathrm{Exp}(0.084)$). Number of estimated parameters is $s = 1$.

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❹ Goodness of Fit.

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  Choose intervals (make E_i equal).

Class Interval	*Observed Frequency* O_i	*Expected Frequency* E_i	$\frac{(O_i - E_i)^2}{E_i}$
[0, 1.590)	19	6.25	26.01
[1.590, 3.425)	10	6.25	2.25
[3.425, 5.595)	3	6.25	0.81
[5.595, 8.252)	6	6.25	0.01
[8.252, 11.677)	1	6.25	4.41
[11.677, 16.503)	1	6.25	4.41
[16.503, 24.755)	4	6.25	0.81
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  So, p-value $= \mathbb{P}(R \geq r) = \mathbb{P}(R \geq 39.6) = 5 \times 10^{-7}$.

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  Hence, at almost any practical level of significance, e.g., $\alpha = 0.05$, $\alpha = 0.01$, we will reject H_0.